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Research Article

The Unique Maximal *GF*-Regular Submodule of a Module

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An R-module A is called GF-regular if, for each $a \in A$ and $r \in R$, there exist $t \in R$ and a positive integer n such that $r^n t r^n a = r^n a$. We proved that each unitary R-module A contains a unique maximal GF-regular submodule, which we denoted by $MGF(A)$. Furthermore, the radical properties of A are investigated; we proved that if A is an R-module and K is a submodule of A, then $MGF(K) = K \cap MGF(A)$. Moreover, if A is projective, then $MGF(A)$ is a G-pure submodule of A and $MGF(A) = M(R) \cdot A$.

1. Introduction

Throughout this paper, R is a commutative ring with identity and all modules are left unitary, unless otherwise stated. Recall that an element $r \in R$ is said to be regular if there exists $t \in R$ such that $rtr = r$; a ring R is called regular if and only if each element of R is regular. An ideal I of a ring R is regular if each of its elements is regular in R; indeed, a regular ideal I of R is itself a regular ring [1]. Brown and McCoy proved in [1] that each ring R contains a unique maximal regular ideal $M(R)$ which satisfies the well-known radical properties. The ideal $M(R)$ is called the regular radical of R.

The concept of regularity was extended to modules in several ways and in [2] the notion of F-regular modules (in the sense of Fieldhouse [3]) was generalized to GF-regular modules. Let A be an R-module; an element $a \in A$ is said to be GF-regular if for each $r \in R$ there exist $t \in R$ and a positive integer n such that $r^n t r^n a = r^n a$. An R-module A is called GF-regular if and only if all its elements are GF-regular; in particular, a ring R is GF-regular if and only if R is GF-regular as an R-module. On the other hand a ring R is π-regular if and only if R is a GF-regular R-module; recall that a ring R is π-regular if, for each $r \in R$, there exist $t \in R$ and a positive integer n such that $r^n t r^n = r^n$. A submodule N of an R-module A is called GF-regular if each element of N is GF-regular and every submodule of a GF-regular module is a GF-regular module. Also, in [2] the concept of G-pure submodules was introduced; a submodule P of an R-module A is called G-pure if, for each $r \in R$, there exists a positive integer n such that $P \cap Rr^n A = Rr^n P$.

In this paper we show that each module contains a unique maximal GF-regular submodule, which we denote by $MGF(A)$, and we show that $MGF(A)$ satisfies some but not all of the usual radical properties.

2. Main Results

Theorem 1. *Let R be any ring. Every R-module contains a unique maximal GF-regular submodule.*

Proof. Let R be any ring, let A be an R-module, and let

$$G = \{N \mid N \text{ is a } GF\text{-regular submodule of } A\}, \quad (1)$$

where $G \neq \phi$ because (0) is a GF-regular submodule of A. Let $\{N_i\}$ be an ascending chain in G and $B = \bigcup_{i\in\Lambda} N_i$. Let $b \in B$; there exists $j \in \Lambda$ such that $b \in N_j$, but N_j is a GF-regular submodule; then; for each $r \in R$, there exist $t \in R$ and a positive integer n such that $r^n t r^n b = r^n b$; therefore b is a GF-regular element in B which implies that B is a GF-regular R-module. Now, by Zorn's lemma, G contains a maximal element which we call MGF. To prove the uniqueness of MGF, assume that $MGF1$ and $MGF2$ be two maximal GF-regular submodules in A; then for any maximal ideal P of R each of $MGF1_p$ and $MGF2_p$ is semisimple over R_p [2, Proposition 21]. Now, let $MGF1_p \cap MGF2_p = K_p$; then $K_p \subseteq MGF1_p$ and $K_p \subseteq MGF2_p$; thus $MGF1_p = K_p + A1_p$ and

$MGF2_p = K_p + A2_p$, where $A1_p$ and $A2_p$ are two submodules of A_p [4]. Hence, $MGF1_p + MGF2_p = A1_p + K_p + A2_p$, but each of $A1_p$, $A2_p$, and K_p is a semisimple submodule; thus $MGF1_p + MGF2_p$ is a semisimple submodule which implies that $MGF1_p + MGF2_p$ is GF-regular [2]. So $MGF1 + MGF2$ is a GF-regular submodule [2, Theorem 20]. Now, each of $MGF1$ and $MGF2$ is a maximal GF-regular submodule and hence $MGF1 + MGF2 = MGF2 = MGF1$. □

Remark 2. We denote the unique maximal GF-regular submodule of an R-module A by $MGF(A)$. It is obvious that $MGF(A)$ contains every GF-regular submodule of A; this means that $MGF(A)$ is a GF-regular submodule which is not contained properly in any other GF-regular submodule. In fact, $MGF(A)$ is the sum of all GF-regular submodules of A and $MGF(A) = A$ if and only if A is a GF-regular module.

Example 3. (a) Since the Z-module Z_n is GF-regular for each positive integer n [2], then $MGF(Z_n) = Z_n$.

(b) Each element in the Z-module Q is not GF-regular [2]; hence $MGF(Q) = (0)$.

(c) Let p be a prime number and let $A = Z_{p^\infty} = \bigcup_{\forall i} Z_{p^i}$ be a Z-module. Let $a \in \bigcup_{\forall i} Z_{p^i}$; then there exists a positive integer m such that $a \in Z_{p^m}$, but Z_n is a GF-regular Z-module for each positive integer n; hence a is a GF-regular element, so $a \in MGF(Z_{p^\infty})$ which implies that $MGF(Z_{p^\infty}) = Z_{p^\infty}$.

(d) Let Q/Z be a Z-module; since Q/Z is a torsion Z-module, then Q/Z is a GF-regular Z-module [2, Proposition 6]. Since $Q/Z = \sum_p Z_{p^\infty}$ for each prime number p, then $MGF(Q/Z) = \sum_p Z_{p^\infty}$ for all primes p.

Proposition 4. *Let A and B be R-modules, and let K be a submodule of A; then*

(a) $MGF(K) = K \cap MGF(A)$,

(b) $MGF(A \oplus B) \subseteq MGF(A) \oplus MGF(B)$.

Proof. (a) Let K be a submodule of A, and let $k \in MGF(K)$; then $k \in K$ and k is GF-regular in K which implies that k is GF-regular in A, thus $k \in K \cap MGF(A)$. Conversely, let $k \in K$ and $k \in MGF(A)$; therefore k is GF-regular in K which means that $k \in MGF(K)$ and hence $MGF(K) = K \cap MGF(A)$.

(b) Let $c \in MGF(A \oplus B)$; then $c = (a, b)$, where $a \in A$ and $b \in B$. Since c is GF-regular, then each of a and b is GF-regular which means that $a \in MGF(A)$ and $b \in MGF(B)$; hence $c \in MGF(A) \oplus MGF(B)$. □

Proposition 5. *Let A and A' be R-modules, and let $f : A \to A'$ be an R-homomorphism; then $f(MGF(A)) \subseteq MGF(f(A))$.*

Proof. If $a \in MGF(A)$, then $R/ann(a)$ is a π-regular ring, but $ann(a) \subseteq ann(f(a))$; thus $R/ann(f(a))$ is an epimorphic image of $R/ann(A)$; hence it is a π-regular ring. Therefore, $f(a) \in MGF(f(a))$ and $f(MGF(A)) \subseteq MGF(f(A))$. □

Remark 6. (a) If $f : A \to A'$ is an R-epimorphism, then $f(MGF(A)) \neq MGF(A')$ in general. In fact, let $\pi : Z \to Z_4 \simeq Z/4Z$ be the natural map, where Z and Z_4 are Z-modules. It is easy to check that $MGF(Z) = (0)$, $f(MGF(Z)) = (0)$, but $MGF(Z_4) \simeq Z_2$.

(b) It is shown in [1] that, for a ring R, $M(R/M(R)) = (0)$ which is not true in case of GF-regular modules; this means that $MGF(A/MGF(A)) \neq (0)$ (as in (a)).

Corollary 7. *For each R-module A, $M(R) \cdot A \subseteq MGF(A)$.*

Proof. For each $a \in A$, let $f : R \to A$ be an R-homomorphism defined by $f(r) = ra$. Then $f(M(R)) \subseteq MGF(A)$ by Proposition 5, but $M(R) \cdot A = \sum_a f(M(R))$; hence $M(R) \cdot A \subseteq MGF(A)$. □

Let $J(R)$ be the Jacobson radical of a ring R. Brown and McCoy proved in [1] that $M(R) \cap J(R) = (0)$. However, this is not true for GF-regular modules; for example, if $A = Z_4$ is a Z-module, then $MGF(A) \simeq Z_2$, $J(A) \simeq Z_2$ and $MGF(A) \cap J(A) \neq (0)$.

Lemma 8. *Let A be an R-module and let P be a G-pure submodule of A. For any $r \in R$, there exists a positive integer n such that $P = Rr^nP$ if and only if $P \subseteq Rr^nA$.*

Proof. Since P is G-pure in A, then, for each $r \in R$, there exists a positive integer n such that $P \cap Rr^nA = Rr^nP$. If $P = Rr^nP$, then $P \cap Rr^nA = P$, and hence $P \subseteq Rr^nA$. Conversely, if $P \subseteq Rr^nA$, then $P \cap Rr^nA = P$, but $P \cap Rr^nA = Rr^nP$; therefore $P = Rr^nP$. □

Lemma 9. *Let $r \in J(R)$; if P is a finitely generated G-pure submodule of an R-module A such that $P \subseteq Rr^nA$ for some positive integer n, then $P = 0$.*

Proof. By Lemma 8 we get that $P = Rr^nP$ and by Nakayama's lemma [5], $P = 0$. □

Theorem 10. *Let A be an R-module. If $MGF(A)$ is a G-pure submodule of A, then $MGF(A) \cap J(R) \cdot A = (0)$.*

Proof. Let $r \in MGF(A) \cap J(R) \cdot A$, and let $P = Rr$. It is clear that $P \subseteq MGF(A)$. Since $MGF(A)$ is a GF-regular module, then P is a G-pure submodule in $MGF(A)$ [2, Theorem 11]. But $MGF(A)$ is G-pure in A; hence P is G-pure in A. Now, $P \subseteq J(R) \cdot A$, so $P = 0$ by Lemma 9. Therefore $MGF(A) \cap J(R) \cdot A = (0)$. □

Recall that $M(R)$ is always a pure ideal in R. Hence $M(R)$ is G-pure [2].

Theorem 11. *Let A be a projective R-module; then*

(a) $MGF(A) = M(R) \cdot A$,

(b) *$MGF(A)$ is a G-pure submodule of A,*

(c) $MGF(A) \cap J(A) = (0)$.

Proof. (a) By the dual basis lemma [4], for each $a \in A$ we have that $a = \sum_i f_i(a)a_i$, where a, $a_i \in A$ for all i and $f_i \in A^* := \mathrm{Hom}_R(A, R)$. If $a \in MGF(A)$, then the submodule Ra is GF-regular and $f_i(Ra)$ is a GF-regular ideal in R by Proposition 5, hence $M(R)$. Thus $MGF(A) \subseteq M(R) \cdot A$. We get the other direction of the inclusion by Corollary 7.

(b) First we claim that; for any two ideals K and L of R, $(K \cap L)A = KA \cap LA$; it is enough to show this locally; thus we may assume that A is free. It is clear that $(K \cap L)A \subseteq KA \cap LA$. On the other hand, let $x \in KA \cap LA$; then

$$x = \sum r_i x_i = \sum s_i x_i \quad r_i \in K, \ s_i \in L. \tag{2}$$

By freeness, $r_i = s_i$ and $x \in (K \cap L)A$. Now, let K be any ideal in R; then by (a) we get $MGF(A) \cap KA = M(R) \cap KA = (M(R) \cap K)A$. But $M(R)$ is G-pure ideal, so, for each $r \in R$, there exists a positive integer n such that $M(R) \cap Rr^n = Rr^n M(R)$; hence $MGF(A) \cap Rr^n A = Rr^n M(R) \cdot A = Rr^n MGF(A)$.

(c) Since A is projective, then $J(A) = J(R) \cdot A$ [4] which implies that $MGF(A) \cap J(A) = (0)$ by Theorem 10. □

Corollary 12. *Let R be any ring, and let A be any R-module such that $MGF(A)$ is a G-pure submodule and $J(A) = J(R) \cdot A$; then $MGF(A) \cap J(A) = (0)$.*

Proof. Since $J(A) = J(R) \cdot A$, then by Theorem 10 we get that $MGF(A) \cap J(A) = (0)$. □

Remark 13. If A is a GF-regular R-module, then $MGF(A) = A$; hence $J(R) \cdot A = J(R) \cdot A \cap A = J(R) \cdot A \cap MGF(A) = (0)$. In fact, this shows that Theorem 10 is a generalization of [2, Proposition 28].

In [2] we noticed that every module over π-regular ring is GF-regular, but the converse need not be true in general. The next result shows how the converse may be true, but first we recall that if A is an R-module, then the trace of A is $\mathrm{tr}(A) = \sum_{f \in A^*} f(A)$, where $A^* = \mathrm{Hom}(A, R)$.

Proposition 14. *Let A be a GF-regular R-module. If $\mathrm{tr}(A) = R$, then R is a π-regular ring.*

Proof. For each $a \in A$ and $f \in A^* = \mathrm{Hom}(A, R)$, since Ra is a GF-regular submodule of A, then by [2, Proposition 7] we get that $f(Ra)$ is a π-regular ideal. Thus $f(Ra) \subseteq M(R)$, but $f(Ra) \subseteq \mathrm{tr}(A)$; hence $\mathrm{tr}(A) = R \subseteq M(R)$, which implies that $M(R) = R$ and R is π-regular. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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